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Hypercyclic Linear Groups (HLG)

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ABSTRACT

There exist no hypercyclic linear groups with paraheight w.

Keywords: Linear Groups; Paraheight Groups; Finite Groups

Introduction and Definitions

Let G be an arbitrary group A a G-module and H the split extension of A by G. We say that A is a hypercyclic G-module with paraheight y if A is H-hypercyclic (as a normal subgroup of H) with H-paraheight y.

Proposition

Let G be a finite group and A a hypercyclic G-module. Then A has paraheight at most W. If A is torsion-free as Z-module A has paraheight at most 1 G 1 and if A has jinite exponent e as Z-module and (e, 1 G 1) = 1 then A has paraheight at most 1 G 1 [logs e].

1. Suppose that A is Z-torsion-free. Embed A in V = Q 0~ A by identifying A and 1 @ A. V is a QG-module containing an ascending series of QG-submodules whose factors have Q-dimension at most 1. Also, V is completely reducible as QG-module (Extension of Maschke's Theorem, Ref. [1-7], IV.8.j) and thus V is a direct sum of irreducible QG-modules of Q-dimension 1. Let VI, V, ,..., V, be the homogeneous components of V as QG-module and put Ai = A n Vi. Suppose that Ai i: (0). If g E G there exists a rational number s such that ag = sa for all a in Ai. Since A is hypercyclic there exists b E A, \(O) such that (b) is a G-submodule of Ai, that is 6g = nb for some integer n. Hence s = n is an integer (since A is torsion-free) and consequently every Z-submodule of Ai is a G-module. Thus, A has paraheight at most Y and clearly Y < / G j.

2. Suppose that A is an elementary abelian p-group where (p, / G I) = 1. By the extension of Maschke's Theorem A is completely reducible as F, G-module. Let A, A, ,..., A, be the homogeneous components of A as lF,G-module. Since A is hypercyclic G acts on each Ai as a group of scalars and thus every Z-submodule of each Ai is a G-submodule. Therefore, A has paraheight at most Y < 1 G I. If A has finite exponent e then A contains a characteristic series of length at most [log, e] whose factors are elementary abelian. Hence by 2. if (e, 1G I) = 1, then A has paraheight at most I G / [log, e].

3. General Case. A contains a free abelian subgroup E such that A/E is a periodic abelian group. Then E1 = fiBEG Eg is a free abelian G-submodule of A and A/E, is still periodic as Z-module (since G is finite). Hence by 1. above we may assume that A is periodic as Z-module. A = 0, A, where A, is the p-primary component of A. If I3 is a G-submodule of A such that every Z-submodule of every primary component of B is G-invariant then every Z-submodule of B is G-invariant. Hence the paraheight of A is equal to the upper bound of the paraheights of the A, . Thus we may suppose that A is a p-group. Let A, be the G-submodule of A generated by all the irreducible G-sub-modules of A of order p and inductively define A,+,/A, = (A/A,), . Put A, = (JB A, . A,+JA, is elementary abelian and completely reducible as lF,G-module. There exists only a finite number (r say) of irreducible lF,G-modules of order p up to isomorphism. Let A,+,(j)/A, , j = 1, 2,..., r, be the homogeneous components A,+JA, as IF,G-module. As in 2. Above every Z-sub-

module of A,+,(j)/A, is G-invariant and thus A, has paraheight at most w. It remains to show that A, = A. If a E A, (aG) is a finite G-submodule of A (since G is finite and A is periodic). Now A is hypercyclic, so there exists a finite series of G-submodules, {0} = B, C B, C **. C B, = (aG) such that (B, : B,-J = p f or each i. Clearly Bi C Ai for each i and thus (aG) C A, . Therefore A = A, and 4.1 is proved. It follows from 4.1 and 3.3 that a locally supersoluble linear group has paraheight less than ~2 and that a locally supersoluble linear group over a finitely generated integral domain is parasoluble (use the existence of a nilpotent subgroup of finite index and for the second part [lo, 4.101). TO obtain the relative case and the bounds in terms of 11we have to a little more work that remarkably parallels the hypercentral situation ([lo] Chap. 8).

Lemma

Let V be a vector space of dimension n over the field F, G a subgroup of AutF(V) and W an FG-submodule of V of F-dimension d. If A is a G-hypercyclic normal subgroup of G stabilizing the series (0) C W 2 V then A has G-paraheight at most d (n - d). Proof. If a E A let a' be the linear mapping of VI W into W given by v + W H v (u - 1). The map 4: a ++ a' is a (group) monomorphism of A into the additive group of Hom, (V/ W, W). The latter is an FG-module, the G action being given byf 9: v + W H (vg-l + W) fg; f E Hom, (V/ W, W) and g E G. A is abelian and so is a G-module via conjugation. A simple check shows that $ is a G-module homomorphism. Let B be the F-subspace of Hom, (V/ W, W) spanned by A+. B is an FG- module and since A is G-hypercyclic contains an ascending series of FG-submodules whose factors have F-dimension at most 1. B has F-dimension at most d (n - d). Thus, there exists a series of FG-submodules (0) =B, C B, C ... C B, = B where each B, +lIB, has F-dimension 1 and r < d (n - d).

Supersoluble Linear Groups 53: Since A$ is G-hypercyclic and spans B there exists c in B, +, /B, \{O} such that (c) is a G-module. Now if g E G there exists 01 in F such that xg = OLXfor all x in B, +, /B, Hence EC = nc for some integer n and thus a=TZl,. Therefore every Z-submodule of B, +,/B, is a G-submodule and so B has paraheight (as G-module) at most Y. Since A sc A+ _CB the result follows.

Corollary

Let G be a subgroup of GL(n,F) and U a unipotent G-hypercyclic normal subgroup of G. Then U has G-paraheight at most +z(n - 1). Proof. Let V be the n-row vector space over F regarded as a G-module in the usual way. U is unitriangularizable over F [IO, I.211 and so there exists a non-trivial subspace of V on which U acts trivially. Let W = C,(U). Then d = dim, W 3 1 and W is G-invariant. By induction on II, UCo(V/W)/Co(V/ W) has G-paraheight at most +(n - d)(n - d - 1). By 4.2 U n C,(V/W) has G-paraheight at most d(n - d). Thus U has G-paraheight at most t(n - d)(n + d - 1) < +n(n -1) since I < d < n.

Lemma

If G is an irreducible subgroup of GL(n, F) then X(G) contains a diagonalizable subgroup A normal in G such that /\(G)/A is isomorphic to a subgroup of S, , the symmetric group on n letters. Proof. Let V be the n-row vector space over F regarded as G-module in the usual way and suppose that {V, , V, ,..., V,} is a minimal system of imprimitivity of V as FG-module. If H = NG(Vf), H acts primitively and irreducibly on Vi ([lo] 1.10). Let + be the induced homomorphism of H into AutF(VJ and put 2 = (H n h(G))+ n &(H$). H n A(G) is an H-hypercyclic subgroup of H. If (H n h(G))+ # 2 there exists an element h of (H n X(G))+\ 2 such that (2, h) is normal in H4. But clearly (2, h) is abelian and so lies in the centre of H$ by Blichfeldt's theorem [10, 1.131. This contradiction shows that H n X(G) acts on Vi as a group of scalars. Let A = X(G) n h No(VJ.i=l Then A is a diagonalizable normal subgroup of G and /\(G)/A is isomorphic to a subgroup of S, (since the elements of G permute the Vi among them-selves).

Main Theorem: If G is a subgroup of GL(n, F) G has paraheight at most w + [log, n!]. If G is a subgroup of GL(n, R) where R is a finitely generated integral domain then G has finite paraheight.

Proof: There exist irreducible representations pr , pa ,..., p,. of G and an x in GL(n, F) such that for all g in G. Let U = & ker pi and put H = {diag(gp, ,..., gp7) : g E G> C GL(n, F). U is a unipotent normal subgroup of G and G/U is isomorphic to the completely reducible subgroup H of GL(n,F). By 4.4 H contains a diagonalizable subgroup A that is normal in H such that h(H)/A is isomorphic to a subgroup of S, . Clearly A(H)/A is H-hypercyclic with H-paraheight at most[log, n!]. By [lo] 1.12 (H : C,(A)) is finite. Hence A has H-paraheight at most w (4.1) and thus H has paraheight at most w + [log, n!]. Since G/ U s H, h(G)/ U n h(G) has G-paraheight at most w + [log, n!] and by 4.3 U n A(G) has G-paraheight at most +z(n - 1). Thus G has paraheight at most w + [logsn!]. Suppose now that G _CGL(n, R). It follows from [8-10] 4.10 that A is finitely generated. Therefore by 4.1 H, and thus G, has finite paraheight. We now consider what paraheights are possible for hypercyclic linear groups of low degree. For each positive integer n and each characteristic p denote by ~(n,p) the upper bound of the paraheights of hypercyclic linear groups of degree n over a field of characteristic p. We shall see that o(n, p) is never a limit ordinal.

Proposition

There exist no hypercyclic linear groups with para-height w. Proof. Let G be a hypercyclic subgroup of GL(n,F) with paraheight w. By induction on n and 4.3 G is irreducible. Hence G contains an abelian normal subgroup A of finite index. G contains a series of normal subgroups (1) = G,cG,C ... c G, = G of length w such that every subgroup of Gi+JG, is abelian and normal in G. Since (G : A) is finite there exists a finite r such that AG, = G. Then G/G, is abelian and so G has paraheight at most r + 1. This contradiction proves the proposition. However, there do exist linear groups with paraheight w. These may be constructed exactly as the example 8.7 of [lo] using examples (e.g. (C's, 1 Cs) constructed below in place of C,, 1 C, .

References

1. Rw carter, b fischer, T Hawkes (1968) Extreme classes of finite soluble groups. J Algebra 285-313.
2. JWS CA~SELS, A FRGHLICH (1967) “Algebraic Number Theory,” Academic Press, New York/London.
3. kw Gruenberg (1966) The Engel structure of linear groups. J Algebra 291-303.
4. M Hall (1959) “The Theory of Groups,” Macmillan Co. New York.
5. VP PLATONOV (1966) The Frattini subgroup of linear groups and finite approxi-mability (Russian), Dokl. Akad. AJauk SSSR 798-801; So&t Math. Dokl, pp. 1557-1560.
6. DJS Robinson (1968) “Infinite Soluble and Nilpotent Groups,” Queen Mary College Mathematics Notes, London.
7. E Schenkman (1965) “Group Theory,” Van Nostrand Co., Princeton.
8. WR Scott (1964) “Group Theory,” Prentice Hall, Englewood Cliffs. NJ.
9. BAF Wehrfritz (1966) “Periodic Linear Groups,” Ph.D. Thesis, University of London.
10. BAF WEHRFRITZ (1969) “Infinite Linear Groups,” Queen Mary College Mathe-matics Notes, London.

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